A Point of inflection problem

Question **(a)** Find the range for which the function: $f(x) = x \sin(\ln x)$, $x > 0$

is **(i)** concave **(ii)** convex ;

(b) Hence find the point(s) of inflection of the function.

Solution

(a) $f'(x) = \sin(\ln x) + x\cos(\ln x) \times \frac{1}{x}$

$= \sin(\ln x) + \cos(\ln x)$ **(eq 1)**

$f''(x) = \frac{\cos(\ln x) - \sin(\ln x)}{x}$ **(eq 2)**

$f'''(x) = -\frac{2\cos(\ln x)}{x^2}$ **(eq 3)**

(i) The function f(x) is concave upwards if $f''(x) \geq 0$. (For strict concavity, we use $f''(x) > 0$.)

$\cos(\ln x) - \sin(\ln x) \geq 0$

$\cos(\ln x) \geq \sin(\ln x)$ (See Note 1 at the end of this article.)

By drawing sine and cosine graphs and noting the intervals, we get:

$$\ln x \in \left[(2n-1)\pi + \frac{\pi}{4},\ 2n\pi + \frac{\pi}{4}\right] \quad , n = 0, \pm 1, \pm 2, \ldots$$

$$x \in \left[e^{(2n-1)\pi + \frac{\pi}{4}},\ e^{2n\pi + \frac{\pi}{4}}\right] \quad , n = 0, \pm 1, \pm 2, \ldots$$

(ii) The function f(x) is convex upwards if $f''(x) \leq 0$.

$\cos(\ln x) - \sin(\ln x) \leq 0$

$\cos(\ln x) \leq \sin(\ln x)$

$$\ln x \in \left[2n\pi + \frac{\pi}{4},\ (2n+1)\pi + \frac{\pi}{4}\right] \quad , n = 0, \pm 1, \pm 2, \ldots$$

$$x \in \left[e^{2n\pi + \frac{\pi}{4}},\ e^{(2n+1)\pi + \frac{\pi}{4}}\right] \quad , n = 0, \pm 1, \pm 2, \ldots$$

(b) $f''(x)$ changes sign as x increases through the following of points.

(1) $x = e^{(2n-1)\pi + \frac{\pi}{4}} \quad , n = 0, \pm 1, \pm 2, \ldots$

(2) $x = e^{2n\pi + \frac{\pi}{4}} \quad , n = 0, \pm 1, \pm 2, \ldots$

There may be two sets of points of inflection:

For the points in Set (1), the function changes from <u>convex upwards</u> to <u>concave upwards</u>.

For the points in Set (2), the function changes from <u>concave upwards</u> to <u>convex upwards</u>.

The function is also well-defined and continuous at the points in both Set (1) and Set (2).

The corresponding values of y can be found by substituting in $f(x) = x \sin(\ln x)$:

(1a) $(x, y) = \left(e^{(2n-1)\pi + \frac{\pi}{4}}, -\frac{1}{\sqrt{2}} e^{(2n-1)\pi + \frac{\pi}{4}} \right), \quad n = 0, \pm 1, \pm 2, \ldots$

(2a) $(x, y) = \left(e^{2n\pi + \frac{\pi}{4}}, \frac{1}{\sqrt{2}} e^{2n\pi + \frac{\pi}{4}} \right), \quad n = 0, \pm 1, \pm 2, \ldots$

Let us investigate the problem more closely.

By substituting both sets into the second derivatives in eq (2), we get:

(1b) $f''\left(e^{(2n-1)\pi + \frac{\pi}{4}} \right) = \dfrac{\cos\left[(2n-1)\pi + \frac{\pi}{4} \right] - \sin\left[(2n-1)\pi + \frac{\pi}{4} \right]}{e^{(2n-1)\pi + \frac{\pi}{4}}} = 0$

(2b) $f''\left(e^{2n\pi + \frac{\pi}{4}} \right) = \dfrac{\cos\left[2n\pi + \frac{\pi}{4} \right] - \sin\left[2n\pi + \frac{\pi}{4} \right]}{e^{2n\pi + \frac{\pi}{4}}} = 0$

$\therefore$ Both set (1) and set (2) give the points of inflection.

The third derivative $f'''(x)$ of eq (3) gives some more insight:

(1c) $f'''\left(e^{(2n-1)\pi + \frac{\pi}{4}} \right) = -\dfrac{2\cos\left[(2n-1)\pi + \frac{\pi}{4} \right]}{\left(e^{(2n-1)\pi + \frac{\pi}{4}} \right)^2} = \dfrac{2}{\sqrt{2} e^{(4n-2)\pi + \frac{\pi}{2}}} > 0$

(2c) $f'''\left(e^{2n\pi + \frac{\pi}{4}} \right) = -\dfrac{2\cos\left[2n\pi + \frac{\pi}{4} \right]}{\left(e^{2n\pi + \frac{\pi}{4}} \right)^2} = -\dfrac{2}{\sqrt{2} e^{4n\pi + \frac{\pi}{2}}} < 0$

Following the result of third derivative test :

(1d) For set (1), the function changes from convex upwards to concave upwards. ($f''(x) \le 0$ to $f''(x) \ge 0$)

(2d) For set (2), the function changes from concave upwards to convex upwards. ($f''(x) \ge 0$ to $f''(x) \le 0$)

Note : **(1)** $\cos(\ln x) \ge \sin(\ln x) \Rightarrow \tan(\ln x) \le 1$ is not correct as $\cos(\ln x)$ can be negative.

(2) By setting $f''(x) = 0 \Rightarrow \cos(\ln x) = \sin(\ln x) \Rightarrow \tan(\ln x) = 1$,

We get $\ln x = m\pi + \frac{\pi}{4}$, $x = e^{m\pi + \frac{\pi}{4}}$ $\quad m = 0, \pm 1, \pm 2, ..$

It is not easy to see that this set can be broken to two different sets of inflectional points.

Here in fact we should take $m = 2n - 1$ and $m = 2n$ cases.